Path Integrals for Radio Astronomy

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Interference

- Interference is a universal phenomenon in physics. In radio astronomy, the scintillation of pulsars and potentially Fast Radio Bursts

- However, multi-dimensional oscillatory integrals are expensive to evaluate numerically

- We present a new integration scheme, based on Cauchy’s integration theorem and Picard-Lefschetz theory

- The algorithm runs in polynomial time, and becomes more efficient as the integrand becomes more oscillatory
Fresnel Integral

- From the path integral to the Fresnel integral

\[
\int_{x(0)=x_s}^{x(1)=x_{obs}} \mathcal{D}x \, e^{iS[x]} = \int dx_\perp e^{\frac{i\omega}{2c} \left[ \frac{(x_\perp - \mu)^2}{d} - \int dz \frac{\omega_p^2 (x_\perp, z)}{\omega^2} \right]}
\]

\[
\frac{1}{d} = \frac{1}{d_{sl}} + \frac{1}{d_{lo}} \quad \omega_p^2 = \frac{n_e(x) e^2}{\epsilon_0 m_e}
\]

- In dimensionless units:

\[
\Psi(\mu, \nu) = \left( \frac{\nu}{\pi} \right)^{D/2} \int d^D x \, e^{i\nu [(x-\mu)^2 + \phi(x)]}
\]

- A multi-dimensional oscillatory integral with the imaginary exponent

\[
\Phi(x) = i\nu \left[ (x - \mu)^2 + \phi(x) \right]
\]
Geometric Optics

- Multi-image regions separated by caustics, where intensity spikes

- In geometric optics, the integral is approximated with the real saddle points of the exponent $\Phi(x)$

- In wave optics, we need to evaluate the integral. Nontrivial behaviour near caustics
Catastrophe theory

\[ \Psi(\mu; \nu) = \left( \frac{\nu}{\pi} \right)^{N/2} \int e^{i \phi(x; \mu)} \nu \, dx \]

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>K</th>
<th>N</th>
<th>( \phi(x; \mu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum/minimum</td>
<td>( A_1^\pm )</td>
<td>0</td>
<td>1</td>
<td>( \pm x^2 )</td>
</tr>
<tr>
<td>Fold</td>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>( x^3/3 + \mu x )</td>
</tr>
<tr>
<td>Cusp</td>
<td>( A_3 )</td>
<td>2</td>
<td>1</td>
<td>( x^4/4 + \mu_2 x^2/2 + \mu_1 x )</td>
</tr>
<tr>
<td>Swallowtail</td>
<td>( A_4 )</td>
<td>3</td>
<td>1</td>
<td>( x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x )</td>
</tr>
<tr>
<td>Butterfly</td>
<td>( A_5 )</td>
<td>4</td>
<td>1</td>
<td>( x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x )</td>
</tr>
<tr>
<td>Elliptic umbilic</td>
<td>( D_4^- )</td>
<td>3</td>
<td>2</td>
<td>( x_1^3 - 3x_1x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2x_1x_2 - \mu_1x_1 )</td>
</tr>
<tr>
<td>Hyperbolic umbilic</td>
<td>( D_4^+ )</td>
<td>3</td>
<td>2</td>
<td>( x_1^3 + x_2^3 - \mu_3x_1x_2 - \mu_2x_2 - \mu_1x_1 )</td>
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Table I: The unfoldings of the seven elementary catastrophes with codimension \( K \leq 4 \), with \( x = (x_1, x_2, \ldots, x_N) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_K) \). The normal forms are defined as the unfolding at parameter \( \mu = 0 \), i.e., \( \phi(x; 0) \).

[Arnol’d 1973, 1975, Berry and Upstill 1980, many others]
Picard-Lefschetz Theory

- Picard-Lefschetz theory: any meromorphic oscillatory integral

\[ I = \int_{\mathbb{R}^D} d^D x \ e^{if(x;\mu)} \]

- can be expressed as a sum of convex integrals

\[ if(x;\mu) = h(x;\mu) + iH(x;\mu) \]

\[ I = \sum_i n_i \int_{\mathcal{J}_i} d^D x \ e^{if(x;\mu)} \]

- with the intersection numbers

\[ n_i = \langle \mathbb{R}^D, \mathcal{K}_i \rangle \]
• The Fresnel integral
\[ \int_{\mathbb{R}} dx \, e^{ix^2} \]
• can be written as a Gaussian integral by a deformation in the complex plane
\[ \frac{1 + i}{\sqrt{2}} \int_{\mathbb{R}} du \, e^{-u^2} \]
• Consider the lens

\[ \phi(x) = \frac{\alpha}{1 + x^2} \]

• Alternatively we can flow the integration domain

\[ \Phi(x, \mu) = h(x, \mu) + iH(x, \mu) \]

\[ \frac{\partial \gamma_\lambda(x_0)}{\partial \lambda} = -\nabla h(\gamma_\lambda(x_0)) \]

\[ \gamma_0(x_0) = x_0 \]

\[ \lim_{\lambda \to \infty} \gamma_\lambda(\mathbb{R}^N) = \mathcal{J} = \sum_i n_i \mathcal{J}_i \]
(a) $\alpha = 2, \mu < -\mu_c$
(b) $\alpha = 2, -\mu_c < \mu < \mu_c$
(c) $\alpha = 2, \mu > \mu_c$
(d) $\alpha = 1, \mu < -\mu_c$
(e) $\alpha = 1, \mu = \mu_c$
(f) $\alpha = 1, \mu > \mu_c$
(g) $\alpha = 1/2, \mu < 0$
(h) $\alpha = 1/2, \mu = 0$
(i) $\alpha = 1/2, \mu > 0$
The Cusp
The Swallowtail
Localized lenses

$$\Psi(\mu) = \int_{\mathbb{R}^2} e^{i\nu[(x-\mu)^2 - \phi(x)]} \, dx$$

$$\phi(x) = \frac{\alpha}{1 + x_1^2 + 2x_2^2}$$
• Two dimensional lens consisting of a blob

\[ \Psi(\mu) = \int_{\mathbb{R}^2} e^{i\nu[(x-\mu)^2-\phi(x)]} dx \]

\[ \phi(x) = \frac{\alpha}{1 + x_1^2 + 2x_2^2} \]

• Caustics:
  • Folds
  • Cusps
  • Hyperbolic
• Complicated lens

\[ \Psi(\mu) = \int_{\mathbb{R}^2} e^{i\nu[(x-\mu)^2 - \phi(x)]} \, dx \]

\[ \phi(x) = \frac{\alpha}{1 + x_1^4 + x_2^2} \]

• Caustics:
  • Folds
  • Cusps
  • Hyperbolic
- Complicated lens

\[ \Psi(\mu) = \int_{\mathbb{R}^2} e^{i\nu[(x-\mu)^2 - \phi(x)]} \, dx \]

\[ \phi(x) = \frac{\alpha(x_1^3 - 3x_1x_2^2)}{1 + x_1^2 + x_2^2} \]

- Caustics:
  - Folds
  - Cusps
  - Elliptic
Summary

• Picard-Lefschetz theory: a new method to evaluate multi-dimensional oscillatory integrals

• A useful tool in wave optics, especially near caustics

• Gravitational microlensing by Dylan Jow.

• Can we detect caustics in scintillation measurements, how much amplifications? [Main et al. 2018]

• Given an interference pattern, can we reconstruct the lens?
Catastrophe theory

$$\Psi(\mu; \nu) = \left(\frac{\nu}{\pi}\right)^{N/2} \int e^{i\phi(x; \mu)\nu} dx$$

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<td>Elliptic umbilic</td>
<td>$D_4^-$</td>
<td>3</td>
<td>2</td>
<td>$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$</td>
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<td>Hyperbolic umbilic</td>
<td>$D_4^+$</td>
<td>3</td>
<td>2</td>
<td>$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$</td>
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[Arnol'd 1973, 1975, Berry and Upstill 1980, many others]
Catastrophe theory

\[ I(0, \nu) = I_0 \nu^{2\beta} \quad \Psi(\mu; \nu) = \nu^\beta \Psi \left( (\nu^{\sigma_1} \mu_1, \ldots, \nu^{\sigma_K} \mu_K), \nu \right) \]

<table>
<thead>
<tr>
<th>Catastrophe</th>
<th>Symbol</th>
<th>( I_0 )</th>
<th>( \beta )</th>
<th>( \sigma_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fold</td>
<td>( A_2 )</td>
<td>1.584</td>
<td>1/6</td>
<td>( \sigma_1 = 2/3 )</td>
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<tr>
<td>Cusp</td>
<td>( A_3 )</td>
<td>2.092</td>
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<td>( \sigma_1 = 3/4, \sigma_2 = 1/2 )</td>
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<td>( \sigma_1 = 4/5, \sigma_2 = 3/5, \sigma_3 = 2/5 )</td>
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<td>1/3</td>
<td>( \sigma_1 = 5/6, \sigma_2 = 2/3, \sigma_3 = 1/2, \sigma_4 = 1/3 )</td>
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<tr>
<td>Elliptic umbilic</td>
<td>( D_4^- )</td>
<td>1.096</td>
<td>1/3</td>
<td>( \sigma_1 = 2/3, \sigma_2 = 2/3, \sigma_3 = 1/3 )</td>
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<td>1/3</td>
<td>( \sigma_1 = 2/3, \sigma_2 = 2/3, \sigma_3 = 1/3 )</td>
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<td>2.258</td>
<td>3/8</td>
<td>( \sigma_1 = 5/8, \sigma_2 = 3/4, \sigma_3 = 1/2, \sigma_4 = 1/4 )</td>
</tr>
</tbody>
</table>

Table II: The intensity and fringe separation scaling relations for the catastrophes shown in Table I. At large \( \nu \) the maximum intensity (14) is given by \( I_0 \nu^{2\beta} \) (see the discussion following Eq. (21)) and the fringe scaling exponents are defined in (22).